C. Field of a moving charge

1. Lienard-Wiechert potential

see Ch. 14 in **JACK.** although he uses relativity. We will follow here a more historical path.

Start with

$$G(\vec{r},t,\vec{r}',t') = \frac{1}{R}\delta(t-t'-R/c) \tag{184}$$

with $\vec{R}=\vec{r}-\vec{r}'$.

Then from

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi = -4\pi k\rho \tag{185}$$

one has

$$\Phi(\vec{r}) = k\int d\vec{r}'\frac{1}{R}\rho(\vec{r}',t_r) \tag{186}$$

with

$$t_r = t - R/c$$

Similarly, from

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} \tag{187}$$

one has

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\vec{r}'\frac{1}{R}\vec{J}(\vec{r}',t_r) \tag{188}$$

Now, for a point charge

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r}_0(t)) \;, \quad \vec{J} = \rho\vec{v}$$

For integration need to replace $\vec{r}$ by $\vec{r}'$ and t by t_r. Note the appearance of the Jacobian in denominator:

$$\partial(\vec{r}' - \vec{r}_0(t_r))/\partial(\vec{r}') = 1 - \frac{\vec{R}\cdot\vec{v}}{Rc}$$

FIG. 19: Evaluation of the potential of a charge moving with constant velocity $\vec{v}$ in the direction OA. Point O is the retraded position, point A is the current position and B is the observation point. $\vec{R}' \equiv \vec{OB}$, $\vec{R} \equiv \vec{AB}$. Note that $\left|\vec{R}'\right|/c = \left|\vec{R}' - \vec{R}\right|/v$

Thus,

$$\Phi = kq\frac{1}{R_r - \vec{v}\cdot\vec{R}_r/c} \;, \quad \vec{A} = \vec{v}\Phi/c^2 \tag{189}$$

(where we indicate retarded position).

2. Charge moving with constant velocity

see Fig. 19.

We use

$$\vec{r}_0(t) = \vec{v}t$$

and

$$\vec{R} = \vec{r} - \vec{r}_0(t) \;, \quad \vec{R}' = \vec{r} - \vec{r}'$$

where $\vec{r}'$ is the retarded position.

From the diagram note

$$OA = OB \cdot v/c$$

or

$$\vec{R}' - \vec{R} = \vec{v}R'/c$$

and

$$R^2 = R'^2 + \frac{v^2}{c^2}R'^2 - 2\frac{\vec{v}\cdot\vec{R}}{c}R' \tag{190}$$

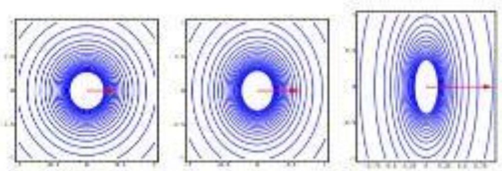

FIG. 20: Potential of a charge moving with a constant velocity, $v/c = 0.4$, 0.6 and 0.9, respectively.

Compare this with the square of denominator in Lienard-Wiechert:

$$\left(R' - \frac{\vec{R}' \cdot \vec{v}}{c}\right)^2 = R^2 - \frac{v^2}{c^2}R'^2 + \frac{1}{c^2}\left(\vec{R}' \cdot \vec{v}\right)^2 = \tag{191}$$
$$R^2 - \frac{1}{c^2}\left(\vec{R}' \times \vec{v}\right)^2 =$$
$$R^2 - \frac{1}{c^2}\left(\vec{R} \times \vec{v}\right)^2$$

Potential is given by

$$\Phi\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2 - \left|\vec{R} \times \vec{v}\right|^2 / c^2}} \tag{192}$$

$$\vec{E}\left(\vec{r}, t\right) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(R^2 - \left|\vec{R} \times \vec{v}\right|^2 / c^2\right)^{3/2}}\frac{\vec{R}}{R^3} \tag{193}$$